Division Algebras, Clifford Algebras, Periodicity

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The dimensions 2, 8 and 24 play significant roles in lattice theory. In Clifford algebra theory there are well-known periodicities of the first two of these dimensions. Using novel representations of the purely Euclidean Clifford algebras over all four of the division algebras, **R**, **C**, **H**, and **O**, a door is opened to a Clifford algebra periodicity of order 24 as well.

Introduction: Bott, Clifford Algebras, Lattices, and Notation

There are well-known periodicities in Clifford algebra (CA) theory of orders 2, 4, and 8 (see [1] for an introduction to Bott periodicity in the CA context, and [2]). These periodicities go hand-in-hand with matrix representations of CAs over the **R** (real numbers), **C** (complex numbers), and **H** (quaternion algebra). In most discussions of CA representations the last division algebra in this sequence (the octonions, **O**), is left out.

In lattice theory the remarkable 24-dimensional Leech lattice ([6]) can be nicely represented in $(\mathbf{O},\mathbf{O},\mathbf{O})$, the 3-dimensional space with octonion components, so 24-dimensional over **R** (see [3], [4], [5]).

Our goal here is to demonstrate that by exploiting the octonion algebra, **O**, in CA representation theory a periodicity of order 24 arises, providing yet another link of the algebra **O** to the dimension 24.

My introduction to the mathematics of both CAs and division algebras - specifically the real numbers **R**, complex numbers **C**, quaternions **H**, and octonions **O** - is [2]. Note: notations have evolved since then, and $\mathcal{CL}(p,q)$ will denote the CA of a p,q-pseudo-orthogonal space with metric signature, p(+), q(-).

I use the following matrix notations:

$$\mathbf{K}(n)$$

the algebra of $n \times n$ matrices over the division algebra **K**.

$$^2\mathbf{K}(n)$$

the block diagonal $2n \times 2n$ matrices over $\mathbf{K}(n)$: (so $2n^2$-dimensional). So, for example, elements of $\mathbf{K}(2n)$ take the form

$$\begin{bmatrix} \mathbf{K}(n) & \mathbf{K}(n) \\ \mathbf{K}(n) & \mathbf{K}(n) \end{bmatrix},$$

and the block diagonal elements of ${}^2\mathbf{K}(n)$ take the form

$$\begin{bmatrix} \mathbf{K}(n) & 0 \\ 0 & \mathbf{K}(n) \end{bmatrix}.$$

In particular, given this basis for $\mathbf{R}(2)$,

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we have this as a basis for ${}^2\mathbf{R}$,

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(All matrices will be dispensed with shortly.)

Further, for any algebra $\mathbf{K}$, let

$$\mathbf{K}_L \text{ and } \mathbf{K}_R \text{ and } \mathbf{K}_A$$

denote the algebras of all actions of $\mathbf{K}$ on itself from the left, the right, and both sides, respectively. In the case of the octonions this requires nested actions due to nonassociativity (see [7] and [5]).

I shall restrict my focus here to the sequences of CAs, $\mathcal{CL}(k,0)$ and $\mathcal{CL}(0,k)$. Since

$$\mathcal{CL}(p+1,q+1) \simeq \mathcal{CL}(p,q) \otimes \mathbf{R}(2),$$

nothing is lost by this restriction of focus (and what I intend to do only works on these ends).

Consider the CA isomorphisms in Table 1 (derived from [2]):

Table 1: Euclidean Clifford Algebra Isomorphisms

k	$\mathcal{CL}(0,k)$		$\mathcal{CL}(k,0)$
0		$\mathbf{R}$	
1	$\mathbf{C}$		${}^2\mathbf{R}$
2	$\mathbf{H}$		$\mathbf{R}(2)$
3	${}^2\mathbf{H}$		$\mathbf{C}(2)$
4		$\mathbf{H}(2)$	
5	$\mathbf{C}(4)$		${}^2\mathbf{H}(2)$
6	$\mathbf{R}(8)$		$\mathbf{H}(4)$
7	${}^2\mathbf{R}(8)$		$\mathbf{C}(8)$
8		$\mathbf{R}(16)$	

Just to clarify,

$$\mathcal{CL}(4,0) \simeq \mathcal{CL}(0,4) \simeq \mathbf{H}(2),$$

so I collapse those two isomorphisms to the center of the table.

Of particular importance,

$$\mathcal{CL}(8,0) \simeq \mathcal{CL}(0,8) \simeq \mathbf{R}(16) \simeq \mathcal{CL}(0,0) \otimes \mathbf{R}(16).$$

This is the first example of Bott periodicity of order 8 in the CA context. In general,

$$\begin{array}{l} \mathcal{CL}(k+8,0) \simeq \mathcal{CL}(k,0) \otimes \mathbf{R}(16), \\ \mathcal{CL}(0,k+8) \simeq \mathcal{CL}(0,k) \otimes \mathbf{R}(16). \end{array}$$

Bott without Matrices

However, we can dispense with all matrix algebras by making use of *split* versions of the division algebras. Bases for $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$ are

$$\begin{array}{l} \mathbf{C}: \ \{1,\ i\}; \\ \mathbf{H}: \ \{q_0 = 1,\ \ q_1,\ \ q_2,\ \ q_3\}; \\ \mathbf{O}: \ \{e_0 := 1,\ \ e_1,\ \ e_2,\ \ e_3,\ \ e_4,\ \ e_5,\ \ e_6,\ \ e_7\} \end{array}$$

$\mathbf{H}$ is noncommutative, but associative, and its multiplication table invariably begins with (and is determined by),

$$q_1 q_2 = -q_2 q_1 = q_3.$$

The multiplication table for $\mathbf{O}$ is determined by specifying bases for 7 quaternionic subalgebras. Specifically, the most elegant of these has quaternionic triples given schematically by the 7 triples,

$$\{e_{1+k}, e_{2+k}, e_{4+k}\},$$

k = 0 to 6, subscripts modulo 7, from 1 to 7. So, set k = 5, yielding the quaternionic triple:

$$e_6 e_7 = -e_7 e_6 = e_2.$$

(See [5] for multiplication tables and much more).

We now need a new copy of the complex algebra, and we'll denote its imaginary unit ι (so $\iota^2 = -1$, and ι commutes with everything, but it is not the same as our original complex unit i). Then bases for split versions of those division algebras (using the multiplication tables above) are

$$\begin{array}{l} \tilde{\mathbf{C}}: \ \{1,\ \iota i\}; \\ \tilde{\mathbf{H}}: \ \{q_0 = 1,\ \ q_1,\ \ \iota q_2,\ \ \iota q_3\}; \\ \tilde{\mathbf{O}}: \ \{e_0 := 1,\ \ e_1,\ \ e_2,\ \ \iota e_3,\ \ e_4,\ \ \iota e_5,\ \ \iota e_6,\ \ \iota e_7\} \end{array}$$

(although these are in fact real algebras, they are no longer division algebras; also, just to be clear, this split version of the octonion algebra requires $\{e_1, e_2, e_4\}$ to be a quaternionic triple, so it should be clear that these bases are not unique in the quaternion and octonion cases; this is not important).

We rid ourselves of all matrix algebras by making use of the following isomorphisms and equivalencies:

$$\begin{gathered}
\tilde{\mathbf{C}} \simeq {}^2\mathbf{R} \\
\mathbf{H} \simeq \mathbf{H}_\mathbf{L} \simeq \mathbf{H}_\mathbf{R} \\
\tilde{\mathbf{H}} \simeq \tilde{\mathbf{H}}_L \simeq \tilde{\mathbf{H}}_R \simeq \mathbf{R}(2) \\
\tilde{\mathbf{H}}^2 \simeq \mathbf{H}^2 \simeq \mathbf{H}_A \simeq \mathbf{R}(4) \\
\mathbf{O}_L = \mathbf{O}_R = \mathbf{O}_A = \tilde{\mathbf{O}}_L = \tilde{\mathbf{O}}_R = \tilde{\mathbf{O}}_A \simeq \mathbf{R}(8)
\end{gathered}$$

In this, and in what follows, it is understood that $\mathbf{K}^n := \mathbf{K} \otimes \mathbf{K} \otimes ... \otimes \mathbf{K}$, where there are n distinct copies of $\mathbf{K}$ on the righthand side (see [7] and [5]).

With these isomorphisms in hand I want to replace the Porteous table of CA isomorphisms above by rewriting it more schematically, using some different isomorphisms, and without matrices:

Table 2: Euclidean Clifford Algebra Isomorphisms without Matrices

				$\mathcal{CL}(0,k)$	k	$\mathcal{CL}(k,0)$				
				$\mathbf{R}$	0	$\mathbf{R}$				
			$\mathbf{C}$	$\mathbf{R}$	1	$\mathbf{R}$	$\tilde{\mathbf{C}}$			
			$\mathbf{H}_\mathbf{L}$	$\mathbf{R}$	2	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H}_\mathbf{L}$	$\mathbf{R}$	3	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$	$\mathbf{R}$	4	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$	$\mathbf{R}$	5	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$	$\tilde{\mathbf{C}}$	
	$\mathbf{H}_\mathbf{L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$	$\mathbf{R}$	6	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H}_\mathbf{L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$	$\mathbf{R}$	7	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_\mathbf{L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{H}_L^4$	8	$\mathbf{H}_L^4$				

Complete by putting $\otimes$ between algebras in $\mathcal{CL}(0,k)$ and $\mathcal{CL}(k,0)$ columns.

We read from this table, for example, that

$$\begin{gathered}
\mathcal{CL}(0,6) \simeq \mathbf{H}_\mathbf{L} \otimes \tilde{\mathbf{H}}_\mathbf{L} \otimes \mathbf{H}_\mathbf{L} \otimes \mathbf{R}, \\
\mathcal{CL}(6,0) \simeq \tilde{\mathbf{H}}_L \otimes \mathbf{H}_L \otimes \tilde{\mathbf{H}}_L \otimes \mathbf{R}.
\end{gathered}$$

In the second line above the pieces of $\mathcal{CL}(6,0)$ are presented in reverse order to highlight the major feature of this table: $\mathcal{CL}(k,0)$ and $\mathcal{CL}(0,k)$ are "split duals" when represented like this. That is, to get $\mathcal{CL}(k,0)$ from $\mathcal{CL}(0,k)$ (or $\mathcal{CL}(0,k)$ from $\mathcal{CL}(k,0)$), replace all its split parts by not split versions (so $\tilde{\mathbf{H}}_L \longrightarrow \mathbf{H}_L$), and replace all not split versions with their split counterparts (so $\mathbf{H}_L \longrightarrow \tilde{\mathbf{H}}_L$).

Of course, these representations are not unique. For example, using the octonion algebra we get

$$\mathcal{CL}(0,6) \simeq \mathbf{O}_L.$$

Interestingly,

$$\tilde{\mathbf{O}}_L \simeq \mathbf{O}_L,$$

so the octonions cannot be exploited in this split duality picture as simply as **C** and **H**, but we shall see that they do have a part to play.

There are some striking periodicities in the table above. Modulo 2 we see that going from $k = 2n$ to $k = 2n + 1$ we alternately add $\mathbf{C}$ or $\tilde{\mathbf{C}}$, which depending on if we are looking at the $\mathcal{CL}(0, k)$ column, or $\mathcal{CL}(k, 0)$. And modulo 4 we see that

$$\mathcal{CL}(0, 4n) \simeq \mathcal{CL}(4n, 0),\ n \geq 0.$$

Modulo 8 is the big periodicity, related to what is known as Bott periodicity. In this context we first see that at $k = 8$ there is a kind of algebraic collapse, or simplification, in the representation. But also,

$$\mathcal{CL}(0, k + 8) \simeq \mathcal{CL}(0, k) \otimes \mathcal{CL}(0, 8),$$
$$\mathcal{CL}(k + 8, 0) \simeq \mathcal{CL}(k, 0) \otimes \mathcal{CL}(0, 8).$$

This kind of order 8 periodicity applies as well to $\mathcal{CL}(p, q)$, with neither p nor q equal to 0, but I'm not interested in that here. However, in that case we lose the split duality. For example, $\mathcal{CL}(1, 1) \simeq \tilde{\mathbf{H}}_L$, which is not self-dual.

Lattices and Dimensions 1, 2, 8 and 24

I accumulated most of my ideas (and what I know) about lattice theory and sphere packings in [5]. My interest in the Leech lattice, specifically, derives from [6], and it relates to my investigations into the roles exceptional mathematical objects, like the division algebras, play in theoretical physics ([7], [5], [10]). Conway and Sloane make it abundantly apparent that the Leech lattice satisfies a great many criteria for exceptionality in this notoriously complex field.

One of the leaders in the field is Henry Cohn who wrote a paper summarizing a recent breakthrough [8]. I'd like to share a few quotes. The initial breakthrough, the work of M. S. Viazovska, related to the sphere packing problem in 8 dimensions [9]. She proved that E_8 (8-dimensional laminated lattice, also denoted Λ_8) is the densest sphere packing in 8 dimensions. Cohn says:

> No proof of optimality had been known for any dimension above three, and Viazovskas paper does not even address four through seven dimensions.

Cohn and collaborators then applied Viazovskas method to prove the Leech lattice (Λ_{24}) is the desnsest packing in 24 dimensions. And again, their work skirts all the intermediate dimensions, 9 to 23. Cohn says:

> Unfortunately, our low-dimensional experience is poor preparation for understanding high-dimensional sphere packing. Based on the first three dimensions, it appears that guessing the optimal packing is easy, but this expectation turns out to be completely false in high dimensions.

> ...
> The sphere packing problem seems to have no simple, systematic solution that works across all dimensions. Instead, each dimension has its own idiosyncracies and charm. Understanding the densest sphere packing in $\mathbf{R}^8$ tells us only a little about $\mathbf{R}^7$ or $\mathbf{R}^9$, and hardly anything about $\mathbf{R}^{10}$.
>
> Aside from $\mathbf{R}^8$ and $\mathbf{R}^{24}$, our ignorance grows as the dimension increases. In high dimensions, we have absolutely no idea how the densest sphere packings behave. We do not know even the most basic facts, such as whether the densest packings should be crystalline or disordered. Here "do not know" does not merely mean "cannot prove," but rather the much stronger "cannot predict."

What's going on here? Why are dimensions 8 and 24 so amenable to proof, and no other high dimensional lattice (none; not one)? The laminated lattices in dimensions 1 and 2 are nice, but the hellish complexity so common in lattice theory begins in dimension 3, and only disappears in dimensions 8 and 24 thereafter.

There are four division algebras associated with parallelizable spheres. These occur in dimensions

$$1,\ 2,\ 4,\ 8.$$

And now we have a new finite sequence of exceptional dimensions revolving around lattice theory:

$$1,\ 2,\ 8,\ 24.$$

Taking these four numbers and dividing by the previous 4, we get

$$1,\ 1,\ 2,\ 3,$$

the beginning of the Fibonacci sequence (I mentioned this stuff in [5]). (This could be mere coincidence, what is in contemporary mathematical parlance referred to as moonshine ([11]).)

One more word about the dimensions 1, 2, 8 and 24. Cohn and Elkies ([8]) developed upper bounds (linear programming bounds) for sphere packings in k dimensions. These bounds vary smoothly, unlike the actual densities of sphere packings that tend to bounce about in a distinctly discontinuous manner. There are four dimensions where the maximal known lattice density in any dimension achieves this upper bound (or appears to to several significant figures): 1, 2, 8 and 24.

Split Dual Clifford Algebra Table up to $k = 24$

Let's take a look at the split dual CA table introduced above, but now expanded to $k = 24$:

Table 3: Clifford algebra isomorphisms to dimension 24.

				$\mathcal{CL}(0,k)$	k	$\mathcal{CL}(k,0)$				
				$\mathbf{R}$	0	$\mathbf{R}$				
			$\mathbf{C}$	$\mathbf{R}$	1	$\mathbf{R}$	$\tilde{\mathbf{C}}$			
			$\mathbf{H_L}$	$\mathbf{R}$	2	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H_L}$	$\mathbf{R}$	3	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	4	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	5	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\tilde{\mathbf{C}}$	
	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	6	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	7	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{H}_L^4$	8	$\mathbf{H}_L^4$				
			$\mathbf{C}$	$\mathbf{H}_L^4$	9	$\mathbf{H}_L^4$	$\tilde{\mathbf{C}}$			
			$\mathbf{H}_L$	$\mathbf{H}_L^4$	10	$\mathbf{H}_L^4$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\mathbf{H}_L^4$	11	$\mathbf{H}_L^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^4$	12	$\mathbf{H}_L^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^4$	13	$\mathbf{H}_L^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{C}}$	
	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^4$	14	$\mathbf{H}_L^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^4$	15	$\mathbf{H}_L^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{H}_L^8$	16	$\mathbf{H}_L^8$				
			$\mathbf{C}$	$\mathbf{H}_L^8$	17	$\mathbf{H}_L^8$	$\tilde{\mathbf{C}}$			
			$\mathbf{H}_L$	$\mathbf{H}_L^8$	18	$\mathbf{H}_L^8$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\mathbf{H}_L^8$	19	$\mathbf{H}_L^8$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^8$	20	$\mathbf{H}_L^8$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^8$	21	$\mathbf{H}_L^8$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{C}}$	
	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^8$	22	$\mathbf{H}_L^8$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_L^8$	23	$\mathbf{H}_L^8$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{O}_L^4$	24	$\mathbf{O}_L^4$				

This table makes the order 8 periodicity very pronounced. At every multiple of 8 there is a kind of algebraic collapse/simplification, after which we start adding things in the same way as we did previously. Keep in mind that few of these representations are unique. For example, at k = 16,

$$\mathbf{H}_A^4 \simeq \mathbf{H}_L^8 \simeq \mathbf{H}_A \otimes \mathbf{O}_L^2.$$

So the octonion algebra could have been introduced before k = 24.

However, 24 is the first dimension for which $\mathcal{CL}(0,k) \simeq \mathcal{CL}(k,0)$, and both can be represented by tensored copies of $\mathbf{O}_L$ (necessarily the same). There is a theme running through this mathematical realm that has arisen elsewhere (see, for example, [5]):

$$\mathbf{H}_L^4 \simeq \mathcal{CL}(8,0) \simeq \mathcal{CL}(0,8),$$

$$\mathbf{O}_L^4 \simeq \mathcal{CL}(24,0) \simeq \mathcal{CL}(0,24).$$

That is, the quaternions are associated with dimension 8, and the octonions with dimension 24.

[Word of explication: 24 is the smallest dimension k for which

$$\mathcal{CL}(k,0) \simeq \mathcal{CL}(0,k),$$

and both can be represented purely in terms of $\mathbf{O}_L$ ($\mathbf{O}_L^4$). The first dimension in which any Clifford algebra can feature the full $\mathbf{O}_L$ in its representation is $n = 6$. And the first dimension in which all $\mathcal{CL}(p,q)$ can exploit $\mathbf{O}_L$ as part of their representations is $p + q = 8$.

$$24 = LCM(6,8).$$

LCM is the least common multiple.]

Let's take a look at a 1-vector basis for the Clifford algebra $\mathcal{CL}(24,0)$ represented by $\mathbf{O}_L^4$. We need four copies of $\mathbf{O}$, and we'll denote their bases by

$${}^m e_a,\ a = 0,...,7,\ \ m = 1,2,3,4.$$

This is the $\mathcal{CL}(24,0)$ 1-vector basis I came up with ($p = 1,...,6$; multiply across rows):

$$\begin{array}{cccc} {}^1e_{Lp} & {}^2e_{L7} & {}^3e_{L0} & {}^4e_{L0} \\ {}^1e_{L0} & {}^2e_{Lp} & {}^3e_{L7} & {}^4e_{L0} \\ {}^1e_{L7} & {}^2e_{L0} & {}^3e_{Lp} & {}^4e_{L0} \\ {}^1e_{L7} & {}^2e_{L7} & {}^3e_{L7} & {}^4e_{Lp} \end{array}$$

This gives us 24 anti-commuting elements of $\mathbf{O}_L^4$ (6 for each row). The product of all 24 is

$$\pm {}^1e_{L7}{}^2e_{L7}{}^3e_{L7}{}^4e_{L7}.$$

Interestingly, if we replace $\mathbf{O}$ by $\mathbf{H}$ (that is, $\mathbf{H}_L^4$), and build a similar basis for a Clifford algebra using quaternions instead of octonions ($r = 1,2$ below), we get

$$\begin{array}{cccc} {}^1q_{Lr} & {}^2q_{L3} & {}^3q_{L0} & {}^4q_{L0} \\ {}^1q_{L0} & {}^2q_{Lr} & {}^3q_{L3} & {}^4q_{L0} \\ {}^1q_{L3} & {}^2q_{L0} & {}^3q_{Lr} & {}^4q_{L0} \\ {}^1q_{L3} & {}^2q_{L3} & {}^3q_{L3} & {}^4q_{Lr} \end{array}$$

which is a basis for $\mathcal{CL}(8,0)$. So the octonions are associated with $\mathcal{CL}(24,0)$, and the quaternions with $\mathcal{CL}(8,0)$, at least within this context.

Lattices, Clifford Algebras, Periodicity

The question is: is this order 24 algebraic collapse to a product of just octonions (left actions) meaningful? It recurs at every dimension $24m$, m a positive integer, so it is periodic.

Topologically Bott periodicity has to do with homotopy groups and the sequences of classical Lie groups, orthogonal, unitary and symplectic. In this context the primary kinds of periodicities that arise are of order 2, 4 and 8.

In the theory of laminated lattices there are also indications of periodicities of order 2, 4, 8 and 24 (see [5]). It was this that inspired this look at Clifford algebra periodicity, and in particular the tantalizing representational collapse $\mathbf{O}_L^4 \simeq \mathcal{CL}(24,0) \simeq \mathcal{CL}(0,24)$. At dimension $24m$ you get the collapse

$$\mathbf{O}_L^{4m} \simeq \mathcal{CL}(24m,0) \simeq \mathcal{CL}(0,24m)$$

so at least in the Clifford algebra context there is an algebraic periodicity of order 24, as well as 8 (which is another manifestation of Bott periodicity).

The question naturally arises: is there a topological periodicity of order 24 associated with this algebraic periodicity (as there is of order 8)? Can this question even be answered given our present mathematical machinery? Is this more than just moonshine? I suggest it is much more.

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