Octonions: E_8 Lattice to Λ_{16}.

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Abstract

I present here another example of a lattice fibration, a discrete version of the highest dimensional Hopf fibration: $S^7 \longrightarrow S^{15} \longrightarrow S^8$.

1. D_4 to E_8 with Quaternions.

To motivate the higher dimensional case involving the octonions, I'll first develop a lower dimensional lattice fibration using the quaternions.

Let $q_m,\ m=0,1,2,3$, be a conventional basis for the quaternion algebra **[1]**, **Q**. Define

$$D_4^+ = \{\pm q_m\} \cup \{\frac{1}{2}(\pm q_0 \pm q_1 \pm q_2 \pm q_3)\}. \tag{1}$$

These $8+16=24$ unit quaternions form the inner shell (nearest neighbors to the origin) of a $D_4=\Lambda_4$ lattice (Λ_k is the real laminated lattice in k dimensions **[2]**). It is well known that the set D_4^+ is closed under multiplication.

Define

$$D_4^- = \{\frac{1}{2}(\pm q_m \pm q_n)\}. \tag{2}$$

This is also the inner shell of a D_4 lattice, these elements normalized to $1/\sqrt{2}$. This is a (shrunken) $Spin(4)$ rotation of D_4^+. However, D_4^- is not closed under multiplication even if expanded to the unit sphere.

From these two sets we can construct the inner shell of the 8-dimensional $E_8=\Lambda_8$ lattice. In particular,

$$\begin{array}{lll} & \{<U,0>,\ <0,V>:\ U,V \in D_4^+\} & (2\times 24 = 48 \text{ elements}) \\ \cup & \{<U,V>:\ U,V \in D_4^-,\ UV^\dagger = \pm q_m\} & (8\times 24 = 192 \text{ elements}) \end{array} \tag{3}$$

($m \in \{0,1,2,3\}$) is the inner shell of an E_8 lattice, a subset of the unit 7-sphere in $\mathbf{Q}^2$.

To illustrate the second of the sets in (3), lets look at the set of all $V \in D_4^-,\ UV^\dagger = \pm q_m$ for $U=\frac{1}{2}(1+q_1)$. There are 8 such elements:

$$\pm\tfrac{1}{2}(1+q_1);\quad \pm\tfrac{1}{2}(1-q_1);\quad \tfrac{1}{2}(\pm q_2 \pm q_3). \tag{4}$$

In fact, there are 8 such elements for each $U \in D_4^-$, hence that second set has $8\times 24 = 192$ elements. The total number of elements is $48+192=240$, which is the order of E_8 **[2]**.

There is another characterization of this 192 element subset:

$$\{<U,V>:\ U,V \in D_4^-,\ V=\pm U \text{ or } U+V \in D_4^+\}. \tag{5}$$

The first two elements in (4) are $\pm U$, and the remaining six elements satisfy $U+V \in D_4^+$.

2. Fibrations.

If $U, V \in \mathbf{Q}$ satisfy

$$UU^\dagger + VV^\dagger = 1,$$

then the doublet $\begin{bmatrix} U \\ V \end{bmatrix}$ is an element of S^7, the (unit) 7-sphere. Define the map

$$\begin{bmatrix} U \\ V \end{bmatrix} \longrightarrow \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}^\dagger = \begin{bmatrix} UU^\dagger & UV^\dagger \\ VU^\dagger & VV^\dagger \end{bmatrix}$$

$$= \tfrac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

$$+\tfrac{UU^\dagger - VV^\dagger}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \tfrac{UV^\dagger + VU^\dagger}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \tfrac{UV^\dagger - VU^\dagger}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The set of all elements

$$< \frac{UU^\dagger - VV^\dagger}{2}, \frac{UV^\dagger + VU^\dagger}{2}, \frac{UV^\dagger - VU^\dagger}{2} >$$

(first two real, third pure quaternion, so 5-dimensional) covers S^4 (the 4-sphere in $\mathbf{R}^5$, in this case of radius $\frac{1}{2}$). The map (6) is an example of the sphere fibration **[1]**

$$S^7 \xrightarrow{S^3} S^4. \tag{7}$$

(Another more interesting example of this fibration in terms of the octonions was given in **[3]**.)

If $\begin{bmatrix} U \\ V \end{bmatrix} \in E_8$, as defined in (3), then the map (6) takes E_8 onto the lattice Z^5 (all inner shells at this point), consisting of elements of the form

$$\tfrac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pm\tfrac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \pm\tfrac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \pm\tfrac{q_i}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{8}$$

($i = 1, 2, 3$). This is an example of the lattice fibration

$$E_8 \xrightarrow{D_4} Z^5. \tag{9}$$

(Again, in **[3]** a more interesting example of this fibration was presented using the octonions.)

3. E_8 to Λ_{16} with Octonions.

Let **O** be the octonion algebra **[1,3,4]**. I choose an octonion multiplication whose quaternionic triples are determined by the cyclic product rule,

$$e_a e_{a+1} = e_{a+5},\ a \in \{1, ..., 7\}, \tag{10}$$

where the indices in (4) are from 1 to 7, modulo 7 (and in particular I will set $7 = 7 \bmod 7$ to avoid confusing e_0 with e_7). This choice, as it turns out, has an influence on what follows **[3,4]**.

Define

$$\begin{aligned} E_8^+ = &\ \{\pm e_a\} \\ \cup &\ \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct},\ e_a(e_b(e_c e_d)) = \pm 1\}, \\ &\ a, b, c, d \in \{0, ..., 7\}. \end{aligned} \tag{11}$$

These $16 + 14 \times 16 = 240$ elements of the unit octonion 7-sphere form the inner shell of an E_8 lattice, which, like D_4^+ is closed under multiplication **[5]**.

Define

$$\begin{aligned} E_8^- = &\ \{\tfrac{1}{2}(\pm e_a \pm e_b) : a, b \text{ distinct}\} \\ \cup &\ \{\tfrac{1}{8}(\textstyle\sum_{a=0}^{7} \pm e_a) : \text{ odd number of +'s}\}, \end{aligned} \tag{12}$$

These $112 + 128 = 240$ elements of the octonion 7-sphere of radius $1/\sqrt{2}$ also form the inner shell of an E_8 lattice, which, like D_4^- is not closed under multiplication. (The effect of my choice of octonion multiplication in (10) is in the definition of E_8^-; there are choices that would require "odd number of +'s" to be changed to "even number of +'s" in (12); this would not change the order of that set, which would still be 128.)

From these two sets we can construct the inner shell of the 16-dimensional Λ_{16} lattice. In particular,

$$\begin{aligned} &\ \{< U, 0 >,\ < 0, V >:\ U, V \in E_8^+\} && (2 \times 240 = 480 \text{ elements}) \\ \cup &\ \{< U, V >:\ U, V \in E_8^-,\ UV^\dagger = \pm e_a\} && (16 \times 240 = 3840 \text{ elements}) \end{aligned} \tag{13}$$

($a \in \{0, ..., 7\}$) is the inner shell of a Λ_{16} lattice, a subset of the unit 15-sphere in $\mathbf{O}^2$.

As an example, let $U = \frac{1}{2}(1 + e_7)$. Then the 16 values of V for which $< U, V > \in \Lambda_{16}$ are:

$$\pm U = \pm\frac{1}{2}(1 + e_7),\ \ \pm\frac{1}{2}(1 - e_7),\ \ \frac{1}{2}(\pm e_1 \pm e_5),\ \ \frac{1}{2}(\pm e_2 \pm e_3),\ \ \frac{1}{2}(\pm e_4 \pm e_6),$$

the last 14 of which satisfy $U + V \in E_8^+$, by which they may also be characterized.

As another example, let $U = \frac{1}{4}(-1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \in E_8^-$.

In this case the 16 appropriate V's are:

$$\begin{aligned}
\pm U = &\pm\tfrac{1}{4}(-1+e_1+e_2+e_3+e_4+e_5+e_6+e_7),\\
&\pm\tfrac{1}{4}(-1+e_1+e_2-e_3-e_4-e_5+e_6-e_7),\\
&\pm\tfrac{1}{4}(-1-e_1+e_2+e_3-e_4-e_5-e_6+e_7),\\
&\pm\tfrac{1}{4}(-1+e_1-e_2+e_3+e_4-e_5-e_6-e_7),\\
&\pm\tfrac{1}{4}(-1-e_1+e_2-e_3+e_4+e_5-e_6-e_7),\\
&\pm\tfrac{1}{4}(-1-e_1-e_2+e_3-e_4+e_5+e_6-e_7),\\
&\pm\tfrac{1}{4}(-1-e_1-e_2-e_3+e_4-e_5+e_6+e_7),\\
&\pm\tfrac{1}{4}(-1+e_1-e_2-e_3-e_4+e_5-e_6+e_7).
\end{aligned}$$

Again, the last 14 of these elements may be characterized by $U+V \in E_8^+$.

4. More Fibrations.

If $U, V \in \mathbf{O}$ satisfy

$$UU^\dagger + VV^\dagger = 1,$$

then the doublet $\begin{bmatrix} U \\ V \end{bmatrix}$ is an element of S^{15}, the (unit) 15-sphere. As before define the map

$$\begin{bmatrix} U \\ V \end{bmatrix} \longrightarrow \begin{bmatrix} U \\ V \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix}^\dagger = \begin{bmatrix} UU^\dagger & UV^\dagger \\ VU^\dagger & VV^\dagger \end{bmatrix}$$

$$= \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{14}$$

$$+\tfrac{UU^\dagger - VV^\dagger}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \tfrac{UV^\dagger + VU^\dagger}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \tfrac{UV^\dagger - VU^\dagger}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The set of all elements

$$< \frac{UU^\dagger - VV^\dagger}{2}, \frac{UV^\dagger + VU^\dagger}{2}, \frac{UV^\dagger - VU^\dagger}{2} >$$

(first two real, third pure octonion, so 9-dimensional) covers S^8 (the 8-sphere in $\mathbf{R}^9$, in this case of radius $\frac{1}{2}$). This is the highest dimensional example of a sphere fibration **[1]**:

$$S^{15} \xrightarrow{S^7} S^8. \tag{15}$$

If $\begin{bmatrix} U \\ V \end{bmatrix} \in \Lambda_{16}$, as defined in (13), then the map (14) takes Λ_{16} onto the lattice Z^9

(inner shells again), consisting of elements of the form

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\pm\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \pm\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \pm\frac{e_a}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{16}$$

($a = 1, ..., 7$). This is another example of a lattice fibration:

$$\Lambda_{16} \xrightarrow{E_8} Z^9. \tag{17}$$

5. Conclusion.

The reader may be curious to know the purpose of this work. If the reader discovers that purpose before I do, I would ask the reader to let me know. For the nonce, it's just pretty stuff.

References

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