Laminated Lattices and Periodicity

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This short and very numerological article stems from my interest in

- the four real, normed division algebras (**R**, **C**, **H** and **O**),
- laminated lattices,
- and Bott periodicity.

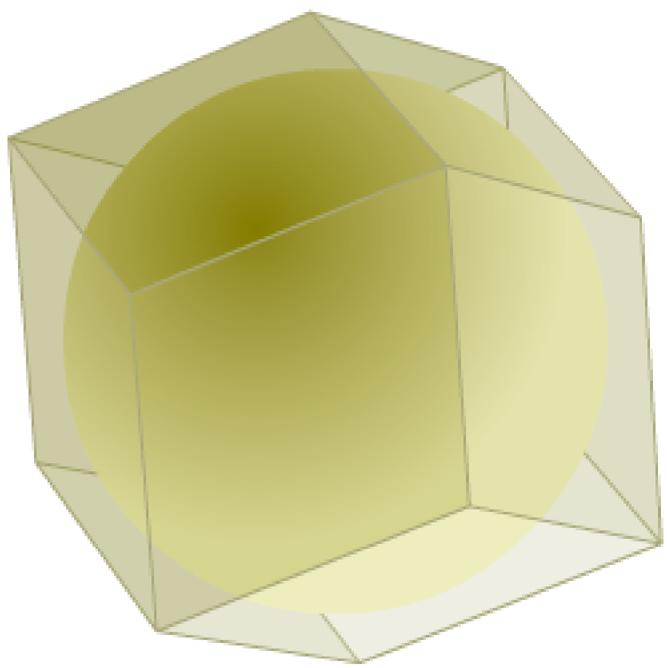
My knowledge of these things diminishes markedly from top to bottom, so this article will not be deep, and I will not be developing much background material beyond what I find personally useful. You're on your own.

The laminated lattice in a k-dimensional Euclidean space is denoted Λ_k. Here is a way to think about building Λ_{k+1} from Λ_k. Consider Λ_1, consisting in this example of the integral points along an x-axis. The nearest (kissing) neighbors to the point at the origin are the points at ±1 (so the kissing number of Λ_1 is $K_1 = 2$). The 'deep holes' of this grouping are the points half way between ±1 and the origin: ±½. At each of these points add 2 extra lattice points. Move one at each location up a distance sin(60°) perpendicularly away from the x-axis, and one down the same distance. The result is a hexagon of points (the inner shell of Λ_2) along with the point at the origin, which, in having six nearest neighbors, implies $K_2 = 6$.

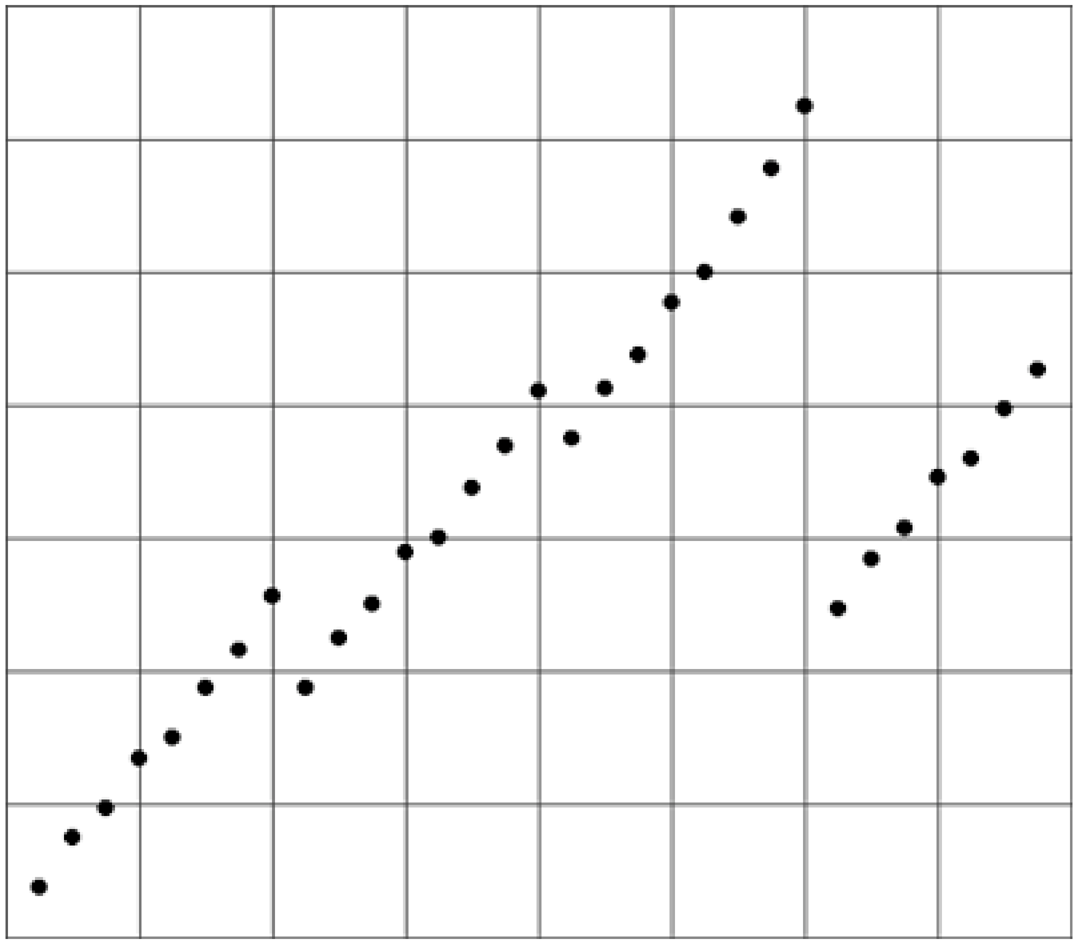
A similar approach can be applied in going from Λ_2 to Λ_3, but things are more complicated now. The deep holes of this arrangement of seven points on the plane are the center points of the six equilateral triangles formed from the origin point and neighboring pairs of inner shell points. In this case we add a single new lattice point at each of these deep holes, and alternately lift or lower these points perpendicularly away from the plane until they form a tetrahedron with the 3 points of the equilateral triangle they were in the center of. The overall result is a truncated cube, with 12 points in the inner shell of Λ_3, so $K_3 = 12$, along with the point in the center.

Let's discuss just one more step, the point of which will be to convince you that whatever you got used to up to some value of k is likely to be different at k+1: induction doesn't function well here.

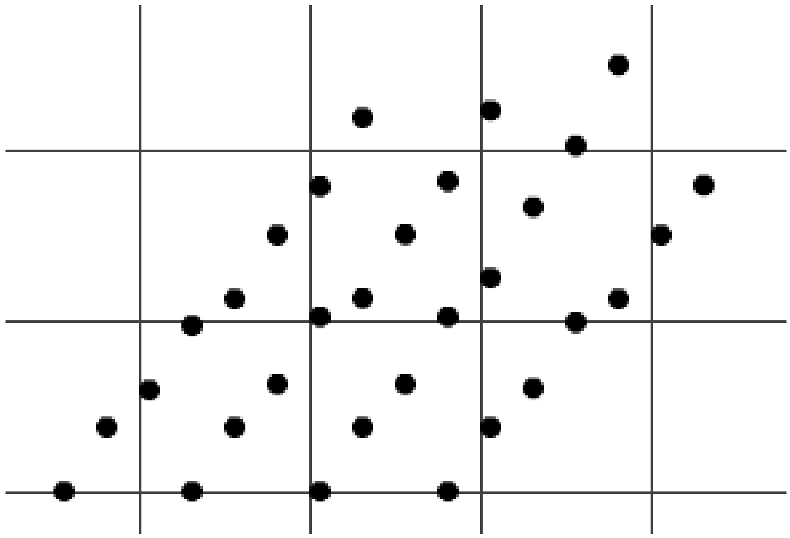
Around each of the 13 points of our inner shell of L_3 and the point of the origin place identical spheres centered at these points with radii just large enough so the spheres touch (kiss). Form tangent planes at each of the 12 kissing points and cut away the excess beyond where these planes intersect. This results in a regular rhombic dodecahedron (rrd). The rrd has two kinds of vertices: where 4 edges meet; and where 3 edges meet. There are 6 vertices of the former kind, and 8 of the latter, and the 4-edge vertices are further from the origin than the 3-edge vertices, and these are the deep holes of the inner shell of our L_3. We now have 8 semi-deep holes, so this situation is more complicated than the 2-d case. We can use these 6 deep holes like we did in the Λ_1 to Λ_2 case to go from Λ_3 to Λ_4. The result is a 4-dimensional lattice with $K_4 = 24$.



So there is a complicated interplay between K_k and all the previous K_{k-n}, and the number of deep holes at each of these values k-n. So, I chose to make a plot of the dimensions k verses $\log_2(K_k - K_{k-1})$, just to see what would happen (see below).



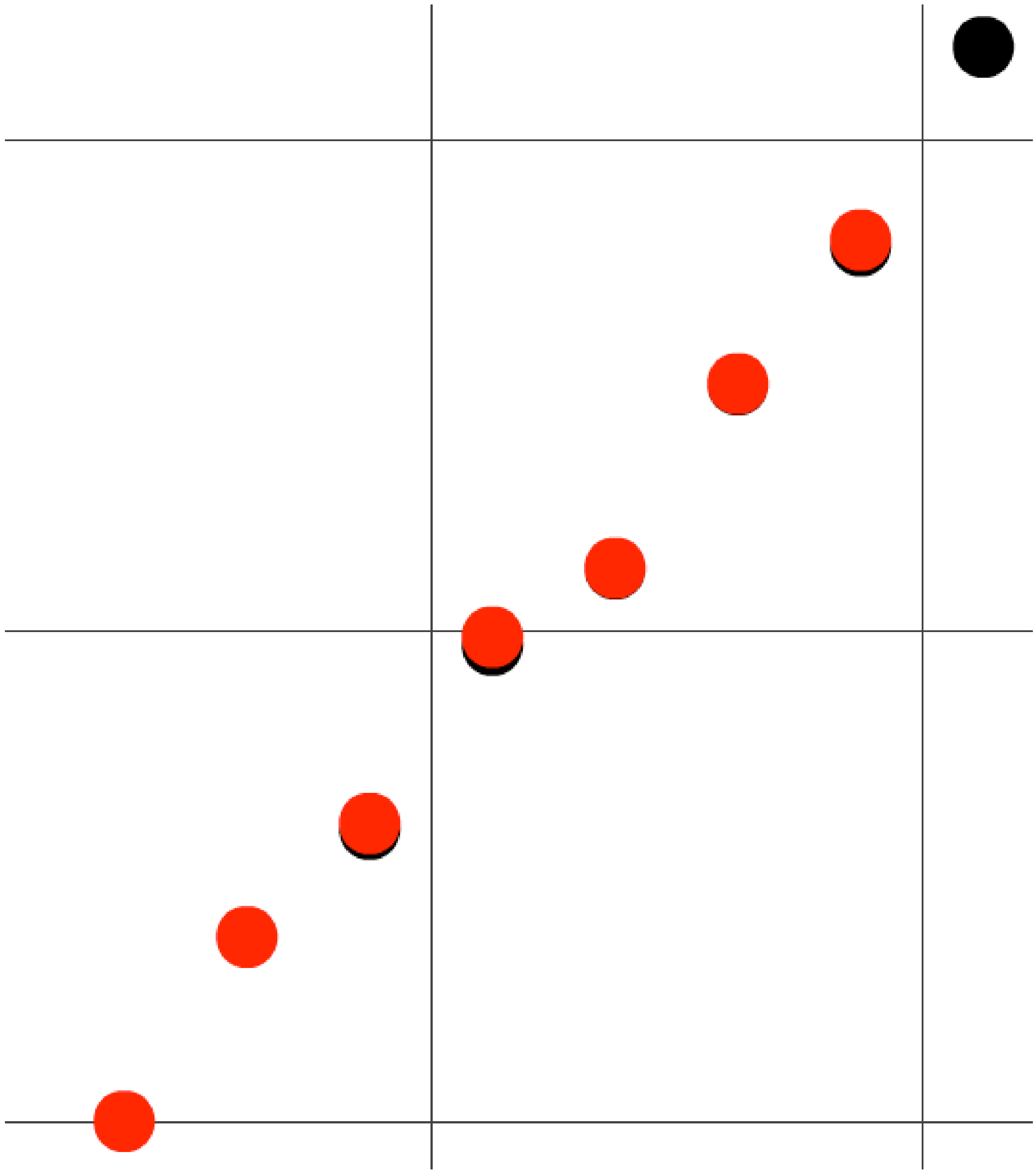
The first thing that pops out of this plot is a pronounced ‘pretty good’ periodicity of order 8, along with evidence for another periodicity of order 24 (note: in may ways Λ_2, Λ_8, and Λ_{24} (Leech lattice) are the most remarkable of this sequence of lattices). And if we put these four clusters of points next to each other we can spy other less pronounced periodicities of orders 2 and 4 (see below).



At this point I wish I understood Bott periodicity better (an order 8 periodicity of various features of k-dimensional Euclidean spaces), because it is obvious that this is related. And it makes one wonder if there is a Bott-like periodicity of order 24. I rather suspect, however, that if there is it will require a different way of looking at geometry to find it.

I'm no longer a serious researcher (one can argue if I ever was), so I've no idea if this is a new idea or not. My sources are primarily Conway & Sloane and Baez and myself.

Final graphic: to highlight how good this order 24 periodicity is, I took the last 7 points, turned them red, and lay them over top the first 8 points, matching the bottom points precisely:



And probably not a final note, as I get the impression I may be updating this article ad infinitum, but this curiosity hit me a couple of days ago. For $k = 2n = 4, 8, 24$ (so $n = 2, 4, 12$):

$K_{2n} = 4n\,(2^n - 1)$;

$K_{2n+1} = 4n\,(2^n + 1)$.

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