Division Algebras: Family Replication

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27 June 2002

The link of the Division Algebras to 10-dimensional spacetime and one leptoquark family is extended to encompass three leptoquark families.

The origins of this work can be found in **[1]**. That book contains Lagrangians, interactions, and in general a more detailed development of the physics resulting from

$$\mathbf{T} = \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$$

than has been presented elsewhwere.

- $\mathbf{C}$ - complex numbers: associative, commutative, basis $\{1, i\}$;
- $\mathbf{H}$ - quaternions: associative, noncommutative, basis $\{1 = q_0, q_1, q_2, q_3\}$;
- $\mathbf{O}$ - octonions: nonassociative, noncommutative, basis $\{1 = e_0, e_1, ..., e_7\}$;

My subsequent work on these division algebras has been largely mathematical. Some of it deals with a more elegant derivation of the Standard Symmetry and lepto-quark family structure than is found in **[1]** (see **[2,3]**). This work accounts neatly for family structure, but it has not until now accounted for family replication.

- $\mathbf{K}_L, \mathbf{K}_R$ - the algebras of left and right actions of an algebra $\mathbf{K}$ on itself.
- $\mathbf{K}_A$ - the algebra of combined left and right actions of an algebra $\mathbf{K}$ on itself.
- $\mathbf{K}$(m) - m×m matrices over the algebra $\mathbf{K}$;
- $\mathbf{K}^m$ - and m×1 column over $\mathbf{K}$;
- $\mathcal{CL}(p,q)$ - the Clifford algebra of the real spacetime with signature (p+,q-).

If we let $\mathbf{P} = \mathbf{C} \otimes \mathbf{H}$, then $\mathbf{P}_L$ is isomorphic to the Pauli algebra, so $\mathbf{P}_L(2)$ is isomorphic to the Dirac algebra, and $\mathbf{H}_R$, which commutes with $\mathbf{P}_L(2)$ (which acts on $\mathbf{H}^2$), provides an internal $SU(2)$ degree of freedom.

One can do much the same thing **[1,2]** with $\mathbf{T}$. $\mathbf{T}_L$ is a Pauli-like algebra, and $\mathbf{T}_L(2)$ is the Dirac algebra of 1,9-spacetime. Again there remains the internal $\mathbf{H}_R$ commuting with $\mathbf{T}_L(2)$, providing an isospin $SU(2)$. The associated spinor space ($\mathbf{T}^2$) transforms with respect to the standard symmetry as the direct sum of a leptoquark family and antifamily of 1,3-Dirac spinors.

But why should we need 2x2 matrices acting on $\mathbf{T}^2$? And where are the other two families? To answer the second question I'll aggravate the first. In particular, we'll assume our spinor space is not just $\mathbf{T}^2$, but

$$\mathbf{C} \otimes \mathbf{H}^2 \otimes \mathbf{O}^3 = \mathbf{T}^6,$$

which is acted upon by $\mathbf{T}_A(6)$. (Octonion triples play important roles in many areas - derivations of the Leech lattice, the exceptional Jordan algebra, triality - which lends support to the idea that this expansion may be natural.)

Obviously, since $\mathbf{T}^2$ accounts for one family/antifamily, $\mathbf{T}^6$ would account for three, which is the accepted number of total families. However, in **[2]** the algebra $\mathbf{T}_L(2)$, which acts on $\mathbf{T}^2$, is isomorphic to a Clifford algebra (the complexification of $\mathcal{CL}(1,9)$). Since all Clifford algebras are 2^k-dimensional, the $3^2 2^{13}$-dimensional $\mathbf{T}_A(6)$ (which is the full algebra of actions associated with $\mathbf{T}^6$) is not a Clifford algebra.

Let's plow ahead anyway, and first look at the 2^{15}-dimensional $\mathbf{T}_A(4)$, isomorphic to the complexification of $\mathcal{CL}(1,13)$. This acts on $\mathbf{T}^4$, which is a pair of leptoquark families (and their antifamilies).

Let q_{Lk} and q_{Rk}, k = 0,1,2,3, be the respective left and right actions of the basis elements of $\mathbf{H}$ on itself. Likewise, e_{La} and e_{Ra}, a = 0,1,...,7, are the same for the octonions, although in this case, since $\mathbf{O}_L = \mathbf{O}_R$, we will not often be using the latter elements. Since the complex numbers are commutative and associative it makes no sense to distinguish left and right actions, so we won't.

Some 2×2 real matrices:

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Define, for example, the following 4×4 real matrix:

$$[\beta \otimes \alpha] = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}.$$

Here is the chosen $\mathcal{CL}(1,13)$ 1-vector basis:

$$[\epsilon \otimes \beta](iq_{R3}),\ \ [\epsilon \otimes \gamma]q_{Lk}e_{L7}(iq_{R3}),\ k = 1,2,3,\ \ [\epsilon \otimes \gamma]ie_{Lp}(iq_{R3}),\ p = 1,...,6,$$

$$[\beta \otimes \epsilon]q_{R1},\ [\beta \otimes \epsilon]q_{R2},\ [\beta \otimes \alpha]q_{R3},\ [\gamma \otimes \alpha].$$

The first line contains 10 elements which generate a $\mathcal{CL}(1,9)$ subalgebra of $\mathcal{CL}(1,13)$. This is essentially the $\mathcal{CL}(1,9)$ that appeared in **[2]**. The second line contains 4 elements which generate a $\mathcal{CL}(0,4)$ subalgebra. Under the commutator product the associated 2-vectors are a basis for $so(4) \sim su(2) \times su(2)$. The six generators are:

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])\{[\epsilon \otimes \alpha]q_{R1},\ [\epsilon \otimes \alpha]q_{R2},\ [\epsilon \otimes \epsilon]q_{R3},\}.$$

The 4×4 real matrix $[\alpha \otimes \epsilon]$ is the product of the last four 1-vectors above, hence it commutes with the $\mathcal{CL}(1,9)$ 1-vectors, but anticommutes with the $\mathcal{CL}(0,4)$ 1-vectors. Therefore it can be used to reduce the 1,13-spacetime to 1,9-spacetime. In particular, at the 1-vector level,

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])\mathcal{CL}(1,13)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]) = \mathcal{CL}(1,9)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]).$$

At the 2-vector level,

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])so(1,13)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]) = (so(1,9) \times su(2))\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]),$$

each projector $\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])$ picking out an $su(2)$ half of $so(4)$, and projecting from the spinor space, $\mathbf{T}^4$, a $\mathbf{T}^2$ subspace. Hence this reduction results in exactly the scenario developed in [**2**], except doubled. Each $\mathbf{T}^2$ subspace is the direct sum of a family and antifamily of leptons and quarks.

With a Clifford algebra and spinors we can form a Dirac operator and Lagrangian. If there were 2^k families then $\mathbf{T}^{2k}$ would be the appropriate hyper-spinor space, acted on by a conventional Clifford algebra. But it is believed there are exactly 3 families, and we will have to get a little creative in constructing a Dirac-like Lagrangian for this case.

A Dirac operator for the $\mathcal{CL}(1,13)$ 2-family model developed above would be

$$\left[\begin{array}{cc} \not{\partial}_{1,9} & \not{\partial}^{+}_{0,4} \\ \not{\partial}^{-}_{0,4} & \not{\partial}_{1,9} \end{array}\right],$$

built from the original set of 14 1-vectors. For the 3-family case, the suggestion is to incorporate 3 of theses 2-family Dirac operators, leading to a Lagrangian term like

$$\left[\begin{array}{ccc} \overline{\psi_1} & \overline{\psi_2} & \overline{\psi_3} \end{array}\right] \left[\begin{array}{lll} \not{\partial}^{a}_{1,9} + \not{\partial}^{b}_{1,9} & \not{\partial}^{a+}_{0,4} & \not{\partial}^{b-}_{0,4} \\ \not{\partial}^{a-}_{0,4} & \not{\partial}^{a}_{1,9} + \not{\partial}^{c}_{1,9} & \not{\partial}^{c+}_{0,4} \\ \not{\partial}^{b+}_{0,4} & \not{\partial}^{c-}_{0,4} & \not{\partial}^{b}_{1,9} + \not{\partial}^{c}_{1,9} \end{array}\right] \left[\begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \end{array}\right]$$

Each of the $\psi_k, k = 1, 2, 3,$ is a complete leptoquark family plus antifamily. On the assumption this approach to a 3-family Lagrangian has some validity many questions arise. Are these the 3 observed families, or mixtures thereof? Are there 3 distinct 14-dimensional spaces? There are many more questions, which my intuition tells me are worth pursuing (no voices - just a gut feeling), but if this happens, it will do so slowly, as I didn't really have time to take it even this far.

References:

[1] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Kluwer, 1994).

[2] G.M. Dixon, www.7stones.com/Homepage/10Dnew.pdf

[3] G.M. Dixon, www.7stones.com/Homepage/octoIII.pdf