HYPERALGEBRA

GEOFFREY DIXON

gdixon@7stones.com

≺ Last modified: 2011.01.09 ≻

The octonion algebra is built from a pair of quaternion algebras. Thoughts on generalizing this to a triple product on a trio of octonion algebras are presented.

Introduction

The dimensions 2, 8, and 24, are the only known dimensions n for which the n-dimensional laminated lattices ($\Lambda_2 = \mathrm{A}_2$, $\Lambda_8 = \mathrm{E}_8$, Λ_{24} = Leech lattice) simultaneously provide the tightest sphere packings, give the best kissing numbers, and the kissing spheres lock into place. Periodicities of orders 2, 8 and 24 occur in the structure of laminated lattices. The suggestion here is that, like the series of dimensions 2^k, $k = 0, 1, 2, 3$, which yield the resonant dimensions (in box),

$$\boxed{1,\quad 2,\quad 4,\quad 8,}\qquad 16,\quad 32,\quad ...,$$

there is another series of dimensions, $F_{k+1}2^k$, $k = 0, 1, 2, 3$, which give rise to the geometrically and algebraically resonant dimensions (in box),

$$\boxed{1,\quad 2,\quad 8,\quad 24,}\qquad 160,\quad 512,\quad ...$$

(F_k is the Fibonacci sequence, 1,1,2,3,5,8,13,21,..., for k = 1,2,3,4,5,6,7,8,.... The use of F_k as coefficients in this formula for this sequence is a guess, bolstered by the apparent coincidence that

$$\prod_1^3 F_k = 2, \prod_1^4 F_k = 6, \prod_1^6 F_k = 240, \prod_1^8 F_k = 196560/3,$$

where 2, 6, 240 and 196560 are the respective kissing numbers of the exceptional laminated lattices, Λ_1, Λ_2, Λ_8, and Λ_{24}.) This goes far to explaining my interest, but this sequence also fits neatly with some work I put together in theoretical physics [**1**], in which the space $\mathbf{T^6} = \mathbf{C} \otimes \mathbf{H^2} \otimes \mathbf{O^3}$ is used to model three generations of lepto-quark families with all the pieces (including righthanded neutrinos) in place. It is my hope that the triple product being sought here, should it be found, may prove useful in that context.

Among those things of interest arising from the dimensions, 1, 2, 4, 8, are the finite sequences of parallelizable spheres and the real normed division algebras. But these geometric and algebraic sequences end at dimension 8, so if the dimension 24 is geometrically and algebraically interesting, it must be so in some new way. As to that, the existence of the Leech lattice already ensures that dimension 24 is geometrically interesting.

The Other Sequence: Number 8

The other sequence is, 1, 2, 8, 24. We begin with the dimension 8 and the 8-dimensional space, $\mathbf{H^2}$. Ordinarily a basis for $\mathbf{H}$ would be represented by elements $q_m, m = 0,1,2,3,$ with q_0 the identity, and $q_1q_2 = -q_2q_1 = q_3$. Then the rest of the multiplication table can be obtained by cycling these three indices modulo 3, from 1 to 3. In the case of the octonions one can construct multiplication tables that are not only invariant under index cycling, but also index doubling (modulo 7). This quaternion product is not invariant under index doubling.

However, there are advantages here to using new indices that give up the cycling invariance and replace it with an index doubling invariance. Our new basis is: $q_m, m = 0,1,2,4,$ where now $q_1q_2 = -q_2q_1 = q_4$, and these products are invariant with respect to index doubling, but modulo 7. Schematically this gives rise to this multiplication table:

	0	1	2	4
0	0	1	2	4
1	1	−0	4	−2
2	2	−4	−0	1
4	4	2	−1	−0

This multiplication table can be thought of as a map from $\mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{H}$.

One can produce a map from

$$\mathbf{H^2} \times \mathbf{H^2} \longrightarrow \mathbf{H^2}$$

that gives $\mathbf{H^2}$ an octonionic structure. That is, the map is the octonion product. For example, using $0,1,2,4,$ for the indices of the first copy of $\mathbf{H}$ in $\mathbf{H^2}$, and $\underline{0}, \underline{1}, \underline{2}, \underline{4},$ for the second, we produce the mapping

	0	1	2	4	7	3	6	5
0	0	1	2	4	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{4}$
1	1	−0	4	−2	$-\underline{1}$	$\underline{0}$	$-\underline{4}$	$\underline{2}$
2	2	−4	−0	1	$-\underline{2}$	$\underline{4}$	$\underline{0}$	$-\underline{1}$
4	4	2	−1	−0	$-\underline{4}$	$-\underline{2}$	$\underline{1}$	$\underline{0}$
7	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{4}$	−0	−1	−2	−4
3	$\underline{1}$	$-\underline{0}$	$-\underline{4}$	$\underline{2}$	1	−0	−4	2
6	$\underline{2}$	$\underline{4}$	$-\underline{0}$	$-\underline{1}$	2	4	−0	−1
5	$\underline{4}$	$-\underline{2}$	$\underline{1}$	$-\underline{0}$	4	−2	1	−0

where the indices on the outside indicate the correspondence to the ordinary octonion indices (a commonly used index cycling and doubling invariant set).

The Other Sequence: Number 24

The point of this article is the following: generalizing from the above, we now want to look for a map, $\mathbf{\Omega}$, taking

$$\mathbf{O}^3 \times \mathbf{O}^3 \times \mathbf{O}^3 \longrightarrow \mathbf{O}^3,$$

the hope being that such a triple product will provide a rich new way to look at 24-space. The multiplication table is in this case not a square array of symbols, but a 3-dimensional cubic array of symbols, so not so easily represented.

To begin with, the elements of $\mathbf{O}^3$ consist of three copies of the octonions, $\mathbf{O}_k, k = 0, 1, 2$, and, generalizing from the quaternion double product above, we'll assume that $\mathbf{\Omega}$ has the following broad general action ($u_k, v_k, w_k \in \mathbf{O}_k$:

$$\mathbf{\Omega}((u_0, u_1, u_2), (v_0, v_1, v_2), (w_0, w_1, w_2)) =$$

$$(\textstyle\sum_{i+j+k=0\%3} \mathbf{\Omega}_0(u_i, v_j, w_k), \sum_{i+j+k=1\%3} \mathbf{\Omega}_1(u_i, v_j, w_k), \sum_{i+j+k=2\%3} \mathbf{\Omega}_2(u_i, v_j, w_k)),$$

where the $\mathbf{\Omega}_k$ (perhaps not distinct) map $\mathbf{O}^3 \longrightarrow \mathbf{O}$.

In the quaternion case there are four 4×4 square arrays of numbers needed to define the map from $\mathbf{H^2} \times \mathbf{H^2} \longrightarrow \mathbf{H^2}$, and each of these arrays is numerically the same, arising from the quaternion product itself (the signs in each array are different).

Again generalizing from the quaternion case, we assume that each of the 27 cubic $8 \times 8 \times 8$ subarrays in this triple product table has the same numbering (different copies of $\mathbf{O}$), and that this numbering arises from the octonion product itself. So, for example, let $e_a, e_b, e_c \in \mathbf{O_0}$, and assume that any product of these three octonions yields another element e_d (to within a sign), then we assume

$$\mathbf{\Omega}_k(e_a, e_b, e_c) = \pm e_d,$$

and that the same numbering will occur even if we choose three elements from different copies of $\mathbf{O_k}$, just as occurred in the quaternion case.

At this point it might seem we are 2/3 done with defining the triple product $\mathbf{\Omega}$, the last step being determining the 24^3 signs in this $24 \times 24 \times 24$ cubic multiplication table. Unfortunately this is not a trivial task. There are 2^{64} ways to sign the 8×8 table we built for the quaternions, but a much smaller number of those will give rise to the octonion division algebra. In this new case, there are 2^{13824} ways of signing the $24 \times 24 \times 24$ cubic multiplication table we have built to this point. Not even a super computer is going to help us find those combinations that yield something truly interesting.

However, things are not quite so dire. For one, it is reasonable to assume that this product, restricted to $\mathbf{O_0}$, can be constructed using the octonion product itself (there are many triple products one can define for $\mathbf{O}$). That being the case, restricting the map $\mathbf{\Omega}$ to $\mathbf{O_0}$, we should expect that the lowest order $8 \times 8 \times 8$ multiplication table cube would be sign invariant with respect to symmetric permutations of the indices, and the $7 \times 7 \times 7$ subcube of nonzero indices would change sign after an antisymmetric permutation. A symmetric permutation corresponds to a 120° rotation of the multiplication table about the main 3-dimensional diagonal, and the antisymmetric permutation would involve a reflection.

Generalizing from the quaternion case developed above it is going to be assumed here that the $23 \times 23 \times 23$ subcube of signs from the entire multiplication table will share these same symmetry properties with respect to the same rotations and reflections.

Further, and again generalizing from the quaternion case, the assumption is going to be made here that this sort of thing will be valid for each of the 27 $7 \times 7 \times 7$ subcubes.

So, for example, let x_a be a basis for O_0, y_a for O_1, and z_a for O_2, then we expect, for example, that ($7 \times 7 \times 7$ rule; indices a, b, c nonzero)

$$\mathbf{\Omega}_k(x_a, y_b, z_c) = \mathbf{\Omega}_k(x_c, y_a, z_b) = -\mathbf{\Omega}_k(x_b, y_a, z_c),$$

and that ($23 \times 23 \times 23$ rule; index of O_0 unit x nonzero)

$$\mathbf{\Omega}_k(x_a, y_b, z_c) = \mathbf{\Omega}_k(z_a, x_b, y_c) = -\mathbf{\Omega}_k(y_a, x_b, z_c).$$

At this point I'd better stop and spend some time making sure these $7 \times 7 \times 7$ and $23 \times 23 \times 23$ suppositions are consistent. My suspicion is that if this whole idea is viable, the nonassociativity of $\mathbf{O}$ will need to be dealt with in some novel way.

Galois Ternary Systems

I'll preface this section with a confession - if it is even necessary. At the time of this writing, the path to my destination is much less clear than the destination itself. So most of what I have written, and am about to write, is part of an evolving whole, and some bits may ultimately not be fit enough to survive. Also, my interest in ternary systems is recent, and even a cursory web search makes it clear that in the last decade or two interest in such systems has grown quite a bit, even in mainstream physics.

A problem with ternary multiplication systems is it is hard not to think of their being built from binary systems, like the octonionic associator, $(xy)z - x(yz)$. However, it is not difficult to construct sets that close under a ternary operation but not binary, and that is what I am looking for in a ternary product for 24-space. An example is in order, first for just 4-space over Z_2.

In **[3]** and elsewhere I looked at generating the division algebras from 2, 4 and 8-dimensional vectors over Z_2, related to Hadamard matrices **[4]**. For example, consider the four binary strings:

$$\begin{aligned} q_0 &= [0,0,0,0], \\ q_1 &= [0,1,0,1], \\ q_2 &= [0,1,1,0], \\ q_3 &= [0,0,1,1]. \end{aligned}$$

Let $+_2$ denote the binary addition of such strings ($0 +_2 0 = 1 +_2 1 = 0$ and $0 +_2 1 = 1 +_2 0 = 1$). Then, for example,

$$q_1 +_2 q_2 = [0 +_2 0, 1 +_2 1, 0 +_2 1, 1 +_2 0] = q_3.$$

The set of q_k closes under the $+_2$ operation, and we can carry on and make of the set a full-fledged quaternion algebra basis, where the 1's and 0's are used to determine the signs of quaternion products (i.e., the binary addition is used to derive a real quaternion product).

Now consider the set

$$\begin{aligned} t_0 &= [1,1,0,0], \\ t_1 &= [0,1,1,0], \\ t_2 &= [0,0,1,1], \\ t_3 &= [1,0,0,1]. \end{aligned}$$

This set does not close under the $+_2$ operation, at least when the operation is performed on pairs of the elements t_k, but it does close on triples. For all $i, j, k \in \{0,1,2,3\}$, there exists $m \in \{0,1,2,3\}$ such that

$$< i, j, k > := t_i +_2 t_j +_2 t_k = t_m.$$

Clearly in holding one of the three indices fixed we can construct binary operations from this ternary operation. For example, define

$$< j, k >_i = < i, j, k >,$$

i fixed. It doesn't make sense to refer to an identity of the triple operation $< i, j, k >$, but the binary operation $< j, k >_i$ does have an identity, which is just t_i itself. So the identity changes depending on which element we fix to achieve our binary operation. And presumably (although I will not pursue this here), in the same way we can achieve the quaternion algebra from the q_i, we can turn $< i, j, k >$ into a full-fledged real triple algebra.

This $< i, j, k >$ ternary system has exactly the properties I feel are necessary. In particular, it is fundamental. The t_i close only if three elements are included in the $+_2$ operation, and although binary operations can be constructed secondarily, the ternary operation is primary.

Another such 4-some can be obtained from the first by adding $u = [1,1,1,1]$ to each 4-some, yielding

$$\begin{aligned} p_0 &= [1,1,1,1], \\ p_1 &= [1,0,1,0], \\ p_2 &= [1,0,0,1], \\ p_3 &= [1,1,0,0]. \end{aligned}$$

Since $u +_2 u +_2 u = u$, and $p_k = q_k +_2 u$, and the collection of q_k is closed under the $+_2$ (even as a ternary operation), the collection of p_k will close under the operation $x +_2 y +_2 z$ (ternary), but not $x +_2 y$.

We can take this further. The elements

$$\begin{aligned} e_0 &= [0,0,0,0,0,0,0,0], \\ e_1 &= [0,1,1,1,0,1,0,0], \\ e_2 &= [0,0,1,1,1,0,1,0], \\ e_3 &= [0,0,0,1,1,1,0,1], \\ e_4 &= [0,1,0,0,1,1,1,0], \\ e_5 &= [0,0,1,0,0,1,1,1], \\ e_6 &= [0,1,0,1,0,0,1,1], \\ e_7 &= [0,1,1,0,1,0,0,1], \end{aligned}$$

can be used to generate the octonion algebra. This set closes under the $+2$ binary operation, but the set of $e_a + v$, where $v = [1,1,1,1,1,1,1,1]$, does not, although it does when applied as a ternary operation.

One can carry on with these ideas, generalizing to Galois fields $GF(p^n)$ for primes $p > 2$. We may come back to that idea.

Some Division Algebra Ternary Systems

Now let

$$\{e_0, e_1, ..., e_7\}$$

be a basis for $\mathbf{O}$, where we'll use the cyclic multiplication determined by

$$e_1 e_2 = e_4.$$

That is, the set $\{e_0, e_1, e_2, e_4\}$ generates a subalgebra of $\mathbf{O}$ isomorphic to $\mathbf{H}$. This set is closed under ordinary binary multiplication. However, the set of basis elements left over,

$$\{e_3, e_5, e_6, e_7\},$$

is not closed with respect to binary multiplication, but is with respect to ternary, but because of nonassociativity we have to specify what we mean by that. Specifically, this set closes under both ternary products $(xy)z$ and $x(yz)$, which are not equivalent. The automorphism group of this 4-dimensional subset of $\mathbf{O}$, with one of these ternary products, is at least $SU(2)$, the subalgebra of G_2 leaving a quaternionic subalgebra of $\mathbf{O}$ invariant.

We can also modify $\mathbf{O}$ with parts of $\mathbf{C}$ and/or $\mathbf{H}$. For example, the set $i\mathbf{O}$ (each element of $\mathbf{O}$ multiplied by the complex imaginary i) is not closed under the usual binary product, but is closed under either of the principal ternary products: $(xy)z$ and $x(yz)$. This is also true of the set spanned by $\{ie_m, m = 0,1,2,4,\ e_k, k = 3,5,6,7\}$, which is i times the split octonions. Many more examples can be concocted by introducing quaternions into the mix.

Arrays

In **[3]** I outlined the notion of a nilpotent Clifford algebra, which can be defined for both pseudo-orthogonal and symplectic bilinear forms. Ordinary Clifford algebras connect to the underlying form via a bilinear anticommutator product on the generators which is linear in the algebra identity. Nilpotent Clifford algebras use a ternary product on the generators, which is linear in these same generators, so the set of generators is closed under this ternary product. Of course, one can also multiply the associated matrices in pairs.

Inspired by work in **[5]** I more recently began to look at expanding on the notion of $n \times n$ matrix algebras, to arrays with more indices. In particular, let

$$\mathcal{A}(m, j, \mathbf{K}) = \text{set of } m \times m \times ... \times m\ \ (m^j) \text{ arrays over } \mathbf{K},$$

where $\mathbf{K}$ is a field or division algebra (or other algebraic set - more on this below).

Let's start by looking at the $27 = 3^3$-dimensional set, $\mathcal{A}(3,3,\mathbf{R})$. Let A, B, C be elements of this set. We require 3 indices taking values in the set $\{0,1,2\}$ to specify the 27 real components of each of these cubic arrays. Generalizing from the $\mathcal{A}(m,2,\mathbf{R})$ case of $m \times m$ real matrices, for which we can define a binary product (matrix multiplication), we define the following ternary product $(: A, B, C :)$ on $\mathcal{A}(3,3,\mathbf{R})$:

$$D_{abc} = \sum_{ijk} A_{aij} B_{ibk} C_{jkc},$$

where $D = \,:\, A, B, C \,:$ is clearly also an element of $\mathcal{A}(3,3,\mathbf{R})$. This is a true ternary product, not based on an underlying binary product (other than the multiplication of real numbers). (Note: in the m^2 case we define the product by summing over a single repeated index. In this m^3 case we have 3 repeated indices. The m^4 case requires 6. You can see where this is going.)

For all positive integers m, $\mathcal{A}(m,2,\mathbf{R})$ is the set of $m \times m$ real matrices. A complete basis for these can be given in terms of symmetric and antisymmetric matrices (with respect to a simple permutation of indices). Things are not so simple in $\mathcal{A}(m,j,\mathbf{R}),\ j > 2$. For example, look at the simplest example, $\mathcal{A}(2,3,\mathbf{R})$. This is an 8-dimensional space of cube arrays. A has 8 components A_{ijk}, where each index takes values 0 or 1. Therefore at least two of them are equal, so there are no nontrivial elements A such that $A_{\rho(ijk)} = -A_{ijk}$ for all indices i, j, k, and for all odd (or even) permutations ρ on the indices.

In $\mathcal{A}(3,3,\mathbf{R})$ there is a 1-dimensional subspace of the full 27-dimensional space the elements of which are antisymmetric with respect to all odd permutations on the indices. Here is a basis:

$$S_{ijk} = 1,\ (ijk) = (012), (201), (120), \quad S_{ijk} = -1,\ (ijk) = (210), (021), (102),$$
$$S_{ijk} = 0 \text{ otherwise}.$$

This element satisfies

$$: S, S, S : = \; -S.$$

Let $\mathcal{V}$ denote the 6-dimensional subspace of $\mathcal{A}(3,3,\mathbf{R})$ consisting of those elements $B \in \mathcal{A}(3,3,\mathbf{R})$ with zero entries except where the indices are distinct. That is, $B_{ijk} \neq 0$ implies $[i:j:k]$ is a permutation of $[0:1:2]$. Therefore, the element S defined above is in $\mathcal{V}$, and we showed that the 1-dimensional subspace spanned by S is a subalgebra of the 27-dimensional trilinear algebra $\mathcal{A}(3,3,\mathbf{R})$.

As it turns out, 6-dimensional $\mathcal{V}$ is also a subalgebra of $\mathcal{A}(3,3,\mathbf{R})$. Let $A, B, C \in \mathcal{V}$, and $D = \, : A, B, C :$. That is, $D_{abc} = \sum_{ijk} A_{aij} B_{ibk} C_{jkc}$. $A \in \mathcal{V}$ implies the indices a, i, j must be distinct for the righthand side to be potentially nonzero. Likewise, $B \in \mathcal{V}$ implies the indices ibk must be distinct, and $C \in \mathcal{V}$ implies the indices j, k, c must be distinct. Ok, so $b \neq i \Longrightarrow b = a$ or $b = j$ (there are only 3 indices to choose from). If $b = a$, then $k = j$, which implies $C_{jkc} = 0$, therefore

$$b = j.$$

$B_{ibk} = B_{ijk}$ can only be nonzero if

$$a = k.$$

And finally, $C_{jkc} = C_{jak}$ can be nonzero only if

$$c = i.$$

Consequently a, b, c are necessarily distinct, and each of the 6 possible nonzero components of D arises from a single product of potentially nonzero components of A, B, and C. Specifically,

$$\begin{aligned} D_{012} &= A_{021} B_{210} C_{102}, \\ D_{201} &= A_{210} B_{102} C_{021}, \\ D_{120} &= A_{102} B_{021} C_{210}, \\ D_{210} &= A_{201} B_{012} C_{120}, \\ D_{021} &= A_{012} B_{120} C_{201}, \\ D_{102} &= A_{120} B_{201} C_{012}. \end{aligned}$$

Note that if $D_{abc} \neq 0$ and $a:b:c$ is a symmetric (antisymmetric) permutation of $0:1:2$, then

$$D_{abc} = A_{acb} B_{cba} C_{bac},$$

where the respective indices of the components of A, B, and C are antisymmetric (symmetric) permutations of $0:1:2$. So $\mathcal{V}$ is a ternary subalgebra of $\mathcal{A}(3,3,\mathbf{R})$, and the subspace of $\mathcal{V}$ spanned by the single element S is a 1-dimensional ternary subalgebra of $\mathcal{V}$. Finally, define

$$\begin{aligned} &R_{ijk} = 1, \; (ijk) = (012), (201), (120), (210), (021), (102), \\ &R_{ijk} = 0 \text{ otherwise.} \end{aligned}$$

Then the subspace spanned by R and S is a 2-dimensional ternary subalgebra of $\mathcal{V}$.

The supposition to this point has been that the 27 components of these ternary algebras have real coefficients. They could also be elements of $\mathbf{C}$, or $\mathbf{H}$ (yielding a 24-dimensional ternary algebra starting from $\mathcal{V}$), or $\mathbf{O}$, although in the latter case we have to introduce parentheses to account for the nonassociativity of $\mathbf{O}$. But the $\mathbf{K}$ in $\mathcal{A}(3,3,\mathbf{K})$ need not be a binary algebra at all. A simple example of this is $\mathcal{A}(3,3,\mathbf{I})$, where $\mathbf{I}$ is the set of all real multiples of the complex unit i. $\mathcal{A}(3,3,\mathbf{I})$ also closes under our ternary matrix product, since $i^3 = -i$.

And in closing, for the time being, there are any number of 4-dimensional selections for $\mathbf{K}$ that would yield 24-dimensional ternary algebras when we start with the 6-dimensional $\mathcal{V}$. It is not known at this point if the Leech lattice could be represented over this space in such a way it would close under the associated ternary product.

$\prec$ Last modified: 2011.01.09 $\succ$

References:

[1] G.M. Dixon, J. Math. Phys. 45, 3878 (2004)

[2] G.M. Dixon, http://www.7stones.com

[3] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Kluwer, 1994 (now Springer-Verlag)).

[4] J.H. Conway, N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, (Springer-Verlag, 2nd Ed., 1991).

[5] A. Dubrovski, V. Guennadi, *Ternary Numbers and Algebras*, arXiv:hep-th/0608073v3.
John Baez, The octonions, Bull. Amer. Math. Soc. 39 (2002), 145-205