Algebraic Spinor Reduction Yields the Standard Symmetry and Family Structure

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An algebraic reduction of 10-d Dirac spinors to 4-d Dirac spinors yields a concomitant reduction of $so(1,9)$ to $so(1,3) \times u(1) \times su(3)$ and lepton-quark family structure. This mathematical result is based on division algebra theory (the internal su(2) arises from the mathematics).

Introduction

The background to this work is developed in elaborate detail in **[1]**, and it will be presented here only schematically. It revolves around the division algebras, their real tensor products, and the algebras of left and right actions of such algebras on themselves. Some familiarity with division algebra theory is assumed. My notation, and some facts, are listed below:

- **O** - octonions: nonassociative, noncommutative, basis $\{1 = e_0, e_1, ..., e_7\}$ (in **[1]** one of a pair of natural symmetric octonion multiplication tables was employed; bowing to the prevailing winds, in this paper the other more commonly used table is employed, one for which $\{e_a,\ e_{a+1},\ e_{a+3}\}$ is a quaternionic triple, a = 1,...,7, indices modulo 7, from 1 to 7);
- **Q** - quaternions: associative, noncommutative, basis $\{1 = q_0, q_1, q_2, q_3\}$;
- **C** - complex numbers: associative, commutative, basis $\{1, i\}$;
- **R** - real numbers.
- $\mathbf{K}_L, \mathbf{K}_R$ - the algebras of left and right actions of an algebra **K** on itself.
- $\mathbf{K}(2)$ - 2x2 matrices over the algebra **K** (to be identified with Clifford algebras);
- $\mathcal{CL}(p,q)$ - the Clifford algebra of the real spacetime with signature (p+,q-);
- ${}^2\mathbf{K}$ - 2x1 matrices over the algebra **K** (to be identified with spinor spaces);
- $\mathbf{O}_L$ and $\mathbf{O}_R$ are identical, isomorphic to $\mathbf{R}(8)$ (8x8 real matrices), 64-dimensional bases are of the form 1, e_{La}, e_{Lab}, e_{Labc}, or 1, e_{Ra}, e_{Rab}, e_{Rabc}, where, for example, if $x \in \mathbf{O}$, then $e_{Lab}[x] \equiv e_a(e_b x)$, and $e_{Rab}[x] \equiv (xe_a)e_b$ (see **[1]**);

- $\mathbf{Q}_L$ and $\mathbf{Q}_R$ are distinct, both isomorphic to $\mathbf{Q}$, bases $\{1 = q_{L0}, q_{L1}, q_{L2}, q_{L3}\}$ and $\{1 = q_{R0}, q_{R1}, q_{R2}, q_{R3}\}$;

- $\mathbf{C}_L$ and $\mathbf{C}_R$ are identical, both isomorphic to $\mathbf{C}$ (so we only need use $\mathbf{C}$ itself);

- $\mathbf{P} = \mathbf{C} \otimes \mathbf{Q}$, 8-dimensional;

- $\mathbf{P}_L = \mathbf{C}_L \otimes \mathbf{Q}_L$, isomorphic to $\mathbf{C}(2) \simeq \mathcal{CL}(3,0) \simeq \mathbf{C} \otimes \mathcal{CL}(0,2)$ ($\mathbf{P}$ is the spinor space of $\mathbf{P}_L$, consisting of a pair of Pauli spinors; the doubling is due to the internal action of $\mathbf{Q}_R$, which commutes with $\mathbf{P}_L$ actions);

- $\mathbf{T} = \mathbf{C} \otimes \mathbf{Q} \otimes \mathbf{O}$, 64-dimensional;

- $\mathbf{T}_L = \mathbf{C}_L \otimes \mathbf{Q}_L \otimes \mathbf{O}_L$, isomorphic to $\mathbf{C}(16) \simeq \mathcal{CL}(0,9) \simeq \mathbf{C} \otimes \mathcal{CL}(0,8)$ (as was the case with $\mathbf{P}$, the algebra $\mathbf{T}$ is the spinor space of $\mathbf{T}_L$, its dimension twice what is expected due to the internal action of $\mathbf{Q}_R$, the only part of $\mathbf{T}_R$ missing from $\mathbf{T}_L$);

- $\mathbf{P}_L(2) \simeq \mathbf{C}(4) \simeq \mathbf{C} \otimes \mathcal{CL}(1,3)$, the Dirac algebra of (1,3)-spacetime (the major difference being that the spinor space, ${}^2\mathbf{P}$, contains an extra internal $SU(2)$ degree of freedom associated with $\mathbf{Q}_R$);

- $\mathbf{T}_L(2) \simeq \mathbf{C}(32) \simeq \mathbf{C} \otimes \mathcal{CL}(1,9)$, the Dirac algebra of (1,9)-spacetime (spinor space ${}^2\mathbf{T}$; one internal $SU(2)$).

Some Lie algebras and their bases:

- $so(7)$ - $\{e_{Lab}$: a,b = 1,...,7$\}$;

- $so(6)$ - $\{e_{Lpq}$: p,q = 1,...,6$\}$;

- LG_2 - $\{e_{Lab} - e_{Lcd}$: $e_a e_b - e_c e_d = 0$, a,b,c,d = 1,...,7$\}$;

- LG_2 explicitly (LG_2 is the 14-d Lie algebra of G_2, the automorphism group of $\mathbf{O}$):

$$
\begin{array}{ll}
e_{L24} - e_{L56}, & e_{L56} - e_{L37}; \\
e_{L35} - e_{L67}, & e_{L67} - e_{L41}; \\
e_{L46} - e_{L71}, & e_{L71} - e_{L52}; \\
e_{L57} - e_{L12}, & e_{L12} - e_{L63}; \\
e_{L61} - e_{L23}, & e_{L23} - e_{L74}; \\
e_{L72} - e_{L34}, & e_{L34} - e_{L15}; \\
e_{L13} - e_{L45}, & e_{L45} - e_{L26};
\end{array}
$$

- $su(3)$ - $\{e_{Lpq} - e_{Lmn}$: $e_p e_q - e_m e_n = 0$, p,q,m,n = 1,...,6$\}$.

Derivation

In general, the subgroup of G_2 leaving some imaginary octonion direction invariant is a copy of $SU(3)$. The $su(3)$ Lie algebra above generates an $SU(3)$ leaving e_7 invariant, a conventional choice for color $SU(3)$. In this case, the projection operators,

$$\rho_\pm = \frac{1}{2}(1 \pm ie_7);\ \rho_{L\pm} = \frac{1}{2}(1 \pm ie_{L7});\ \rho_{R\pm} = \frac{1}{2}(1 \pm ie_{R7}); \tag{1}$$

(note that $\rho_\pm^2 = \rho_\pm,\ \rho_+\rho_- = 0;\ \rho_+ + \rho_- = 1$) can be used to reduce the spinor space ${}^2\mathbf{T}$ into its $SU(3)$ multiplets:

$$\begin{aligned}
\rho_+({}^2\mathbf{T})\rho_+ &= \rho_{L+}\rho_{R+}[{}^2\mathbf{T}] = \text{singlet } (\mathbf{1});\\
\rho_+({}^2\mathbf{T})\rho_- &= \rho_{L+}\rho_{R-}[{}^2\mathbf{T}] = \text{triplet } (\mathbf{3});\\
\rho_-({}^2\mathbf{T})\rho_+ &= \rho_{L-}\rho_{R+}[{}^2\mathbf{T}] = \text{antitriplet } (\bar{\mathbf{3}});\\
\rho_-({}^2\mathbf{T})\rho_- &= \rho_{L-}\rho_{R-}[{}^2\mathbf{T}] = \text{antisinglet } (\bar{\mathbf{1}}).
\end{aligned}$$

Recall that ${}^2\mathbf{T}$ is an $SU(2)$ doublet of (1,9)-Dirac spinors, and we see that each (1,9)-Dirac spinor resolves to $\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \bar{\mathbf{1}}$ with respect to the chosen $SU(3)$, each $SU(3)$ vector a doublet of (1,3)-Dirac spinors.

In particular the singlet,

$$\rho_+({}^2\mathbf{T})\rho_+ = \rho_{L+}\rho_{R+}[{}^2\mathbf{T}],$$

is an $SU(2)$ doublet of (1,3)-Dirac spinors. That is, the projection operator $\rho_{L+}\rho_{R+}$ reduces (1,9)-Dirac spinors to (1,3)-Dirac spinors. The question is:

What effect does the corresponding action have on the (1,9)-Dirac algebra which acts on the spinors?

My way of addressing this problem is to give the (1,9)-Dirac algebra, $\mathbf{T}_L(2)$, an explicit representation, and then apply the action. And in particular, if the spinor action is ${}^2\mathbf{T} \longrightarrow \rho_{L+}\rho_{R+}[{}^2\mathbf{T}]$, then the corresponding action on the (1,9)-Dirac algebra is:

$$\mathbf{T}_L(2) \longrightarrow \rho_{R+}\rho_{L+}\mathbf{T}_L(2)\rho_{L+}\rho_{R+} \tag{2}$$

(note: the projection operators $\rho_\pm$ are Hermitian).

Let α, β, γ, be anticommuting matrices in $\mathbf{R}(2)$ satisfying,

$$\alpha^2 = \beta^2 = -\gamma^2 = \epsilon, \tag{3}$$

where ϵ is the $\mathbf{R}(2)$ identity matrix. I'll use the following representation for the (1,9)-Dirac 1-vectors (the final result is representation independent):

$$\{\beta,\ e_{L7}q_{Lk}\gamma,\ ie_{Lp}\gamma:\ \ k = 1,2,3,\ p = 1,...,6\}. \tag{4}$$

Note:

$$\rho_{L\pm}e_{L7}\rho_{L\pm} = e_{L7}\rho_{L\pm}\rho_{L\pm} = e_{L7}\rho_{L\pm} = \mp i\rho_{L\pm};$$
$$\rho_{L\pm}e_{Lp}\rho_{L\pm} = e_{Lp}\rho_{L\mp}\rho_{L\pm} = 0.$$

Therefore, the action (2) on the 1-vectors (4) is:

$$\text{(1,9)-Dirac 1-vectors} \longrightarrow \{\beta, -iq_{Lk}\gamma\}\rho_{L+}\rho_{R+} \tag{5}$$

(the ρ_{R+} play no part in the 1-vector reduction). What's left is a basis for a representation of (1,3)-Dirac 1-vectors.

> **The $\rho_{L+}\rho_{R+}$ that survive the reduction imply that this set of operators acts only on the $SU(3)$ singlet. Each of the spinor reductions, $\rho_{L\pm}\rho_{R\pm}[^2\mathbf{T}]$, gives rise to the same reduction on the (1,9)-Dirac algebra, save for signs, which may be associated with charges, and the presence of $\rho_{L\pm}\rho_{R\pm}$ surviving the reduction. These operators associate the reduction with the corresponding $SU(3)$ multiplet, and in fact the reduced algebra kills all multiplets but that with it is associated.**

The 2-vector basis arising from the 1-vector basis above is:

$$\{e_{L7}q_{Lk}\alpha,\ ie_{Lp}\alpha,\ q_{Lk}\epsilon\ (so(3)),\ e_{Lpq}\ (so(6)),\ iq_{Lk}e_{Lp7}\epsilon,\ p,q=1,...,6\}. \tag{6}$$

The space of 2-vectors is closed under the commutator product and isomorphic to the Lie algebra $so(1,9)$, the (1,9)-Lorentz group. Some subalgebras are indicated.

The first step in the reduction is finding $\rho_{L+}(so(1,9))\rho_{L+}$. Using results like those at the top of this page we find

$$\rho_{L+}(so(1,9))\rho_{L+} \longrightarrow \{-iq_{Lk}\alpha,\ q_{Lk}\epsilon,\ e_{Lpq}\epsilon\}\rho_{L+}. \tag{7}$$

The result is $so(1,3)\times so(6)$, and significantly the result is not just $so(1,3)$, which is all that would arise from anticommutators of the reduced 1-vectors. Ths $so(6)$ is extra.

Finally we bracket this result with ρ_{R+}. This projector commutes with the $so(1,3)$ part, so it leaves it unaltered (although introducing a ρ_{R+} into the result). So in particular we need to look at

$$\rho_{R+}e_{Lpq}\rho_{R+}. \tag{8}$$

We'll look at some examples, with $e_pe_q \neq e_7$, then $e_pe_q = e_7$, which will give us the general result. Consider $\rho_{R+}e_{L12}\rho_{R+}$. Because ρ_{L+} commutes with $e_{Lpq}, p, q \neq 7$, $\rho_{L+}e_{Lpq}\rho_{L+} = \rho_{L+}e_{Lpq}$. But ρ_{R+} does not commute with these e_{Lpq}. To see what it does we'll re-express our chosen element e_{L12} as

$$e_{L12} = \frac{1}{2}(-e_{R12} + e_{R4} + e_{R63} + e_{R57}) \tag{9}$$

(see **[1]**). ρ_{R+} commutes with e_{R12} and e_{R63}, but becomes ρ_{R-} when commuted with e_{R4} and e_{R57} (recall: $\rho_{R+}\rho_{R-} = 0$). Therefore,

$$\rho_{R+}e_{L12}\rho_{R+} = \frac{1}{2}\rho_{R+}(-e_{R12} + e_{R63}) = \frac{1}{2}\rho_{R+}(e_{L12} - e_{L63}) \tag{10}$$

(see **[1]** for the last equality).

Finally we need to look at the three terms e_{Lpq} for which $e_p e_q = e_7$. These can be re-expressed:

$$\begin{aligned} e_{L13} &= \tfrac{1}{2}(-e_{R13} + e_{R7} + e_{R26} + e_{R45}), \\ e_{L26} &= \tfrac{1}{2}(+e_{R13} + e_{R7} - e_{R26} + e_{R45}), \\ e_{L45} &= \tfrac{1}{2}(+e_{R13} + e_{R7} + e_{R26} - e_{R45}). \end{aligned} \tag{11}$$

All terms in (12) commute with ρ_{R+}, so

$$\rho_{R+}\{e_{L13},\, e_{L26},\, e_{L45}\}\rho_{R+} = \rho_{R+}\{e_{L13},\, e_{L26},\, e_{L45}\}. \tag{12}$$

Another basis for the space spanned by the set $\rho_{R+}\{e_{L13},\, e_{L26},\, e_{L45}\}$ is:

$$\rho_{R+}\{e_{L13} - e_{L26},\; e_{L26} - e_{L45},\; e_{L13} + e_{L26} + e_{L45}\}. \tag{13}$$

But $e_{R7} = \frac{1}{2}(-e_{L7} + e_{L13} + e_{R26} + e_{R45})$, so

$$e_{L13} + e_{L26} + e_{L45} = e_{L7} + 2e_{R7}, \tag{14}$$

which commutes with all the other surviving elements, and therefore generates a copy of $U(1)$. The remaining 8 elements are a basis for $su(3)$.

The central result of this paper is:

$$\rho_{R+}\rho_{L+}so(1,9)\rho_{L+}\rho_{R+} = (so(1,3) \times u(1) \times su(3))\rho_{L+}\rho_{R+} \tag{15}$$

(note: the $u(1)$ charges are tied to the $su(3)$ charges).

We get essentially the same thing if we replace $\rho_{L+}\rho_{R+}$ by any combination $\rho_{L\pm}\rho_{R\pm}$. Let's look at the $u(1)$'s. Using $e_{L7}\rho_{L\pm} = \mp i\rho_{L\pm}$, and $e_{R7}\rho_{R\pm} = \mp i\rho_{R\pm}$, we find

$$\begin{array}{ll} \text{(lepton)} & \frac{1}{3}(e_{L7} + 2e_{R7})\rho_{L+}\rho_{R+} = \frac{1}{3}(-i - 2i)\rho_{L+}\rho_{R+} = -i\rho_{L+}\rho_{R+}; \\ \text{(quark)} & \frac{1}{3}(e_{L7} + 2e_{R7})\rho_{L+}\rho_{R-} = \frac{1}{3}(-i + 2i)\rho_{L+}\rho_{R+} = \frac{1}{3}i\rho_{L+}\rho_{R+}; \\ \text{(antiquark)} & \frac{1}{3}(e_{L7} + 2e_{R7})\rho_{L-}\rho_{R+} = \frac{1}{3}(i - 2i)\rho_{L+}\rho_{R+} = -\frac{1}{3}i\rho_{L+}\rho_{R+}; \\ \text{(antilepton)} & \frac{1}{3}(e_{L7} + 2e_{R7})\rho_{L-}\rho_{R-} = \frac{1}{3}(i + 2i)\rho_{L+}\rho_{R+} = i\rho_{L+}\rho_{R+} \end{array} \tag{16}$$

(factor $\frac{1}{3}$ thrown in to align values with observed hypercharges and conventional $su(3)$ charges). We see that the $u(1)$ charges are linked to the $su(3)$ multiplets.

So each of these projections gives fundamentally the same result:

$$so(1,9) \times su(2) \longrightarrow so(1,3) \times (u(1) \times su(2) \times su(3)) \quad (17)$$

(I've included the $su(2)$ arising from $\mathbf{Q}_R$). The differences are manifested in the multiplet structure of the projected spinors, the charges, and the orientation of 3-space (matter and antimatter are inverted). Note that $u(1) \times su(3)$ is a subalgebra of $so(1,9)$ but commutes with $so(1,3)$, so it is an internal symmetry; $su(2)$ stands alone in all cases.

Speaking of $su(2)$, the reduced spinors are $su(2)$ doublets. Additional projection operators can project from these their individual $su(2)$ components. This reduces $su(2)$ to $u(1)$, which in combination with the $u(1)$ we already have leaves us with the $u(1)$ associated with electric charge. See **[1]** for details.

References:

[1] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Kluwer, 1994).

[2] G.M. Dixon, www.7stones.com/Homepage/history.html